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RADIAL EXPANSION OF HOLLOW SPHERES OF ELASTIC-PLASTIC HARDENING MATERIAL

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Abstract—The problems of expansion of a hollow sphere, or of a spherical cavity in an infinite medium, are treated for homogeneous, isotropic, incompressible, elastic-plastic hardening materials. Five special material response laws, for which the solutions for the applied pressure or tension, and the stress distribution, can be represented in terms of elementary functions, are set down. Four of them involve three parameter plastic hardening laws and the fifth involves a two parameter hardening law. Important features of the solutions for compressible materials are obtained from the solutions for incompressible materials. Analysis of the condition for the occurrence of stationary values of the applied load shows that three types of behavior are possible. The pressure or tension may increase monotonically, or it may increase and then decrease, or it may increase, then decrease, and then increase again.

NOTATION

a Inner radius of hollow sphere
a_0 Initial value of a
a_1 Value of a at initial yield
a_2 Value of a at total yield
A Constant in hardening law
b Outer radius of hollow sphere
b_0 Initial value of b
b_1 Value of b at initial yield
b_2 Value of b at total yield
c Radius of elastic-plastic interface
C_i Constants ($i = 1, \ldots, 5$)
E Young's modulus
f Response function
f' Derivative of f
f^e Elastic response function
f^p Plastic response function
g Response function
g' Derivative of g
h Uniaxial compressive stress response function
h^e Elastic compressive stress response function
h^p Plastic compressive stress response function
H Uniaxial compressive stress response function
H^e Elastic compressive stress response function
H^p Plastic compressive stress response function
K Bulk modulus
n Positive integer
P Applied pressure or allround tension
P_1 Value of P at initial yield
P_2 Value of P at total yield
P^* Maximum value of P
$\bar{P}$ Volume average of hydrostatic pressure
$\mathcal{P}$ Pressure-porosity function
r Radial coordinate in spherical system
r_0 Initial value of r
R Region of space
t_i Cartesian components of surface traction
V Volume of the region R
x Measure of stretch ($x = \lambda^{3/2}$)
x_a Value of x at $r = a$
x_b Value of x at $r = b$
x_r Value of x at r
$\hat{x}$ Value of x at yield
x_i Cartesian coordinates
Y Initial yield strength
Y_i Constants in hardening laws ($i = 1, \ldots, 5$)
Y_∞ Ultimate strength
α Measure of porosity; $\alpha = b^3/(b^3 - a^3)$

α_0 Initial value of α
α_1 Value of α at initial yield
α_2 Value of α at total yield
α^* Value of α at pressure maximum
α^{**} Value of α at pressure minimum
$\bar{\alpha}$ Particular value of α
β Constant in hardening law
γ Constant in hardening law
∂R Boundary of the region R
ΔV Change in volume V
ϵ Compressive logarithmic strain ($\epsilon = -\ln \lambda$)
ϵ_a Value of ϵ at $r = a$
$\hat{\epsilon}$ Yield strain
$\bar{\epsilon}$ Particular value of ϵ
ϵ^+ Particular value of ϵ
ϵ_i Particular values of ϵ ($i = 1, \ldots, 4$)
θ Spherical polar coordinate
θ_0 Initial value of θ
κ Constant in hardening law
$\bar{\kappa}$ Particular value of κ
λ Axial or radial stretch
λ_a Value of λ at $r = a$
λ_b Value of λ at $r = b$
λ_r Value of λ at r
$\hat{\lambda}$ Value of λ at yield ($\hat{\lambda} = e^{-\hat{\epsilon}}$)
μ Shear modulus
σ Compressive stress
σ_a Value of σ at $r = a$
σ_b Value of σ at $r = b$
σ_{rr} Radial component of stress
$\sigma_{\theta\theta}$ Tangential component of stress
$\sigma_{\phi\phi}$ Azimuthal component of stress
σ_{ij} Cartesian components of stress
$\bar{\sigma}_{ij}$ Cartesian components of average stress
ϕ Spherical polar coordinate
ϕ_0 Initial value of ϕ
ψ Material constant ($\psi = Y/2\mu$)

1. INTRODUCTION

The problem of expansion or compaction of hollow spheres of elastic-plastic material, or of spherical cavities in elastic-plastic media of infinite extent, has received considerable attention. Quasistatic solutions of these problems for both elastic-perfectly plastic materials and hardening materials have been given by [1] and by [2]. These results are particularly useful in micromechanical modelling of the response of porous metals, similar to the model[3] for elastic materials. Thus, [4] used the solution for compaction of a rigid-perfectly plastic hollow sphere to model the compaction behavior of metal powders, and [5] used the dynamic solution for an elastic-perfectly plastic hollow sphere under time-varying external pressure to obtain a rate-dependent compaction equation for porous metals.

Solutions of the spherical expansion or compaction problems for strain hardening materials usually involve integrals which must be evaluated numerically. Recently, Carroll and Kim[6, 7] obtained compaction equations for metal powders by solving the compaction problem for a hollow sphere of incompressible, rigid-plastic hardening material, in closed form. Their approach was to choose hardening laws which allow evaluation of the compaction integral in terms of elementary functions.

The problem of the expansion of a hollow sphere of homogeneous, isotropic, incompressible, elastic-plastic hardening material under uniform internal pressure or external allround tension is treated in the present study. The assumption of incompressibility leads to a considerable simplification, since the form of the deformation field is known *ab initio*. In fact, the inner radius at any time determines the entire deformation, so that the deformation field is described by a single parameter. Also, because of incompressibility, the relevant material response property is the function which relates the stress to the axial strain for monotonic uniaxial compressive stress loading.

The applied pressure and the stress distribution are first represented in terms of a single function f, which is defined by a first order ordinary differential equation involving the compressive stress response function. This equation helps to identify the class of elastic and plastic response laws for which solutions of the spherical expansion problem can be expressed in terms of elementary functions. Five such elastic-plastic response laws are adopted. Four of them involve four material constants (the elastic shear modulus, the initial yield strength, and two hardening parameters) and the fifth involves three material constants. They describe either saturation hardening or hardening with unbounded ultimate strength, and they should provide good approximations to actual elastic-plastic hardening data over a wide range of strain.

For each of these response laws, integration of the ordinary differential equation gives the solution function f as a combination of elementary functions. This gives the pressure and the stress distribution in closed form. Solutions are also given for expansion of a spherical cavity in a medium of infinite extent.

The solution of the spherical expansion problem for incompressible elastic-plastic response can be used to obtain the most important features of the solution for compressible materials. This is done for both internal pressure and external allround tension.

An interesting feature of the solutions is the occurrence of a maximum of the applied pressure (or allround tension). Hill[8] observed that the behavior subsequent to the pressure maximum is unstable and corresponds to necking and bursting of the sphere. This instability may have important micromechanical implications with regard to rupture and crack propagation involving nucleation, growth, and coalescence of voids in tensil stress fields. Carroll[9] derived a condition for the occurrence of stationary values of the applied pressure in spherical expansion, for incompressible elastic materials. The same condition applies for elastic-plastic materials with the general hardening law†. The derivation of this condition is sketched for general material response and its implications with respect to the special hardening laws for examined. Realistic compressive stress response allows for three types of behavior of the pressure in inflation of a hollow sphere. The pressure may (A) increase monotonically, or (B) increase to a maximum value and then decrease (or remain constant), or (C) increase monotonically (for sufficiently thick-walled spheres) and increase, decrease, then increase again (for sufficiently thin-walled spheres). Behavior of type B is most typical of elastic-plastic response. The pressure maximum instability is the only type treated here; the possibility of bifurcation instabilities is not examined.

It should be emphasized that the analysis in this study is not based on any particular formulation of the theory of elastic-plastic response, nor does it involve any kinematical approximations. It is not necessary, for instance, to distinguish between work hardening and strain hardening. Given any three-dimensional finite strain theory, it is only necessary to compute the uniaxial compressive stress response function. This then determines the behavior in spherical expansion.

The methods used here for the spherical problem can be used to solve the problem of the expansion of hollow cylinders or cylindrical cavities for materials with special elastic-plastic material response.

2. THE MATERIAL RESPONSE

We first consider the response of a homogeneous, isotropic, incompressible, elastic-plastic material under monotonic uniaxial compressive stress. We write the relationship between the compressive Cauchy or true stress σ and the axial compressive

† Since the relevant response property is the uniaxial compressive stress response, and since unloading is not treated, the solutions obtained here pertain to nonlinearly elastic solids. However, an elastic phase and a plastic phase of the loading function are defined, instead of a strain energy function, and the radius of the elastic-plastic interface is found as part of the solution.

logarithmic strain ϵ as

$$\sigma = H(\epsilon) = \begin{cases} H^e(\epsilon) & (0 \leqslant \epsilon \leqslant \hat{\epsilon}) \\ H^p(\epsilon) & (\hat{\epsilon} \leqslant \epsilon < \infty), \end{cases} \tag{2.1}$$

where $\hat{\epsilon}$ denotes the elastic limit or yield strain. Although we will not treat unloading, it is convenient to introduce the notations H^e and H^p for the elastic and plastic portions of the response function.

The response law (2.1) may also be written in terms of the stretch λ—the ratio of deformed length to undeformed length—thus

$$\sigma = h(\lambda) = \begin{cases} h^e(\lambda) & (1 \geqslant \lambda \geqslant \hat{\lambda}) \\ h^p(\lambda) & (\hat{\lambda} \geqslant \lambda > 0), \end{cases} \tag{2.2}$$

with

$$\epsilon = -\ln \lambda; \qquad \lambda = e^{-\epsilon}. \tag{2.3}$$

The functions H and h are related through

$$H(\epsilon) = h(e^{-\epsilon}); \qquad h(\lambda) = H(-\ln \lambda). \tag{2.4}$$

The initial yield strength Y is given by

$$Y = H^e(\hat{\epsilon}) = H^p(\hat{\epsilon}) = h^e(\hat{\lambda}) = h^p(\hat{\lambda}). \tag{2.5}$$

The obvious implication of incompressibility is that the lateral logarithmic strain is $-\frac{1}{2}\epsilon$. We also assume that the material response is unaltered by superposition of a uniform hydrostatic pressure.

3. VOLUME-PRESERVING EXPANSION OF A HOLLOW SPHERE

A volume-preserving, radially symmetric deformation of a hollow sphere is described by

$$r^3 - r_0^3 = a^3 - a_0^3, \qquad \theta = \theta_0, \qquad \phi = \phi_0, \tag{3.1}$$

where (r, θ, ϕ) are spherical polar coordinates of a typical particle, a is the inner radius of the sphere, and the subscript 0 denotes initial values. We assume that the cavity expands, so that $a \geqslant a_0$. Then the local deformation is a radial *contraction*, with stretch λ_r given by

$$\lambda_r = \frac{\mathrm{d}r}{\mathrm{d}r_0} = \frac{r_0^2}{r^2} \tag{3.2}$$

and with equal lateral stretches $1/\sqrt{\lambda_r}$. The local Cauchy or true stress state is a compressive radial stress $\sigma = \sigma_{\theta\theta} - \sigma_{rr}$ superposed on a hydrostatic pressure $-\sigma_{\theta\theta}$. The hydrostatic pressure has no effect, because of incompressibility†. Thus, the relevant material property is the response under uniaxial compressive stress.

The first equation (3.1) gives, in particular,

$$b^3 - b_0^3 = a^3 - a_0^3, \tag{3.3}$$

† We are excluding a class of materials which are incompressible but have a plastic response which depends on the mean stress, e.g. Drucker-Prager or Mohr-Coulomb conditions.

where b is the outer radius of the sphere. It is evident from eqn (3.1) that the volume-preserving radial expansion is a one parameter deformation, in the sense that the value of the inner radius a at any time completely determines the current deformation. The state of deformation can be described equally well by a nondimensional parameter α, with initial value α_0, which is a measure of the porosity of the sphere and is defined by

$$\alpha = b^3/(b^3 - a^3), \qquad \alpha_0 = b_0^3/(b_0^3 - a_0^3). \tag{3.4}$$

The radial stretches λ_a and λ_b at the inner and outer boundary are given by

$$\lambda_a^{3/2} = a_0^3/a^3 = (\alpha_0 - 1)/(\alpha - 1); \qquad \lambda_b^{3/2} = b_0^3/b^3 = \alpha_0/\alpha. \tag{3.5}$$

The radial equation of equilibrium is

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0. \tag{3.6}$$

If the inflation is brought about by pressurization of the inner boundary, then the boundary conditions are

$$\sigma_{rr} = -P \quad \text{at} \quad r = a; \qquad \sigma_{rr} = 0 \quad \text{at} \quad r = b. \tag{3.7}$$

The material response law is

$$\sigma_{\theta\theta} - \sigma_{rr} = h(\lambda_r), \tag{3.8}$$

with λ_r given by eqn (3.2). Equations (3.6)–(3.8) give

$$\begin{aligned} \sigma_{rr} &= -P + 2\int_a^r h(\lambda_r)\frac{dr}{r} \\ \sigma_{\theta\theta} &= \sigma_{\phi\phi} = \sigma_{rr} + h(\lambda_r) \\ P &= 2\int_a^b h(\lambda_r)\frac{dr}{r}. \end{aligned} \tag{3.9}$$

The change of variable

$$x_r = \lambda_r^{3/2} = 1 - (a^3 - a_0^3)/r^3 = 1 - \frac{a_0^3(\alpha - \alpha_0)}{r^3(\alpha_0 - 1)} \tag{3.10}$$

gives a more convenient form of the solution

$$\begin{aligned} \sigma_{rr} &= -P + \frac{2}{3}\int_{x_a}^{x_r} h(x^{2/3})\frac{dx}{1 - x} \\ \sigma_{\theta\theta} &= \sigma_{\phi\phi} = \sigma_{rr} + h(x_r^{2/3}) \\ P &= \frac{2}{3}\int_{x_a}^{x_b} h(x^{2/3})\frac{dx}{1 - x}, \end{aligned} \tag{3.11}$$

with

$$x_a = a_0^3/a^3 = (\alpha_0 - 1)/(\alpha - 1); \qquad x_b = b_0^3/b^3 = \alpha_0/\alpha. \tag{3.12}$$

Now, let $f(x)$ be a function whose derivative $f'(x)$ is given by

$$f'(x) = \frac{2}{3} h(x^{2/3})/(1 - x). \tag{3.13}$$

The solution (3.11) may be written

$$\begin{aligned} \sigma_{rr} &= -P + f(x_r) - f(x_a) \\ \sigma_{\theta\theta} &= \sigma_{\phi\phi} = \sigma_{rr} + h(x_r^{2/3}) \\ P &= f(x_b) - f(x_a). \end{aligned} \tag{3.14}$$

The initial response of the hollow sphere is elastic. Yielding begins at the inner boundary when the stretch λ_a reaches the value $\hat{\lambda}$. It follows from eqns (3.5) and (3.10) that initial yield occurs at porosity α_1, given by

$$\frac{\alpha_0 - 1}{\alpha_1 - 1} = \hat{x} = \hat{\lambda}^{3/2} = e^{-3\hat{\epsilon}/2}, \tag{3.15}$$

i.e. by

$$\alpha_1 = 1 + (\alpha_0 - 1)/\hat{x} = 1 + (\alpha_0 - 1)e^{3\hat{\epsilon}/2}. \tag{3.16}$$

As the pressure increases, a spherical elastic-plastic interface propagates outward. The radius c of this interface is found from eqn (3.10), thus

$$c^3 = \frac{a^3 - a_0^3}{1 - \hat{x}} = \frac{a_0^3(\alpha - \alpha_0)}{(\alpha_0 - 1)(1 - \hat{x})}. \tag{3.17}$$

Eventually, this interface reaches the outer boundary. The porosity α_2 corresponding to the onset of total yield is given by eqns (3.5) and (3.10) as

$$\alpha_2 = \alpha_0/\hat{x} = \alpha_0 e^{3\hat{\epsilon}/2}. \tag{3.18}$$

The values (a_1, b_1) and (a_2, b_2) of the inner and outer radii at initial yield and total yield may be read off from eqns (3.5), (3.16) and (3.18).

The solution for the pressure in the third equation (3.11) may now be written

$$P = \begin{cases} \dfrac{2}{3}\displaystyle\int_{x_a}^{x_b} h^e(x^{2/3})\frac{dx}{1 - x} & (\alpha_0 \leq \alpha \leq \alpha_1) \\ \dfrac{2}{3}\displaystyle\int_{x_a}^{\hat{x}} h^p(x^{2/3})\frac{dx}{1 - x} + \dfrac{2}{3}\displaystyle\int_{\hat{x}}^{x_b} h^e(x^{2/3})\frac{dx}{1 - x} & (\alpha_1 \leq \alpha \leq \alpha_2) \\ \dfrac{2}{3}\displaystyle\int_{x_a}^{x_b} h^p(x^{2/3})\frac{dx}{1 - x} & (\alpha_2 \leq \alpha < \infty). \end{cases} \tag{3.19}$$

It is evident from eqns (3.13) and (3.19) that it is convenient to write

$$f(x) = \begin{cases} f^e(x) & (1 \geq x \geq \hat{x}) \\ f^p(x) & (\hat{x} \geq x > 0), \end{cases} \tag{3.20}$$

with

$$f^e(\hat{x}) = f^p(\hat{x}). \tag{3.21}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f^e(x) &= \frac{2}{3} h^e(x^{2/3})/(1 - x)\\ \frac{\mathrm{d}}{\mathrm{d}x} f^p(x) &= \frac{2}{3} h^p(x^{2/3})/(1 - x).\end{aligned} \tag{3.22}$$

The solution (3.19) may be written as

$$P = \begin{cases} f^e(x_b) - f^e(x_a) & (\alpha_0 \leqslant \alpha \leqslant \alpha_1)\\ f^e(x_b) - f^p(x_a) & (\alpha_1 \leqslant \alpha \leqslant \alpha_2)\\ f^p(x_b) - f^p(x_a) & (\alpha_2 \leqslant \alpha < \infty),\end{cases} \tag{3.23}$$

and the stress distributions in the elastic and plastic regions may be read off from the first two equations (3.14) by replacing the functions f and h by f^e and h^e and by f^p and h^p, respectively.

If the spherical expansion is brought about by application of a uniform external allround tension P, then the boundary conditions are

$$\sigma_{rr} = 0 \quad \text{at} \quad r = a; \qquad \sigma_{rr} = P \quad \text{at} \quad r = b. \tag{3.24}$$

The expression for the allround tension P is the same as that given above (in the third equation (3.9), for instance) but the stress distribution must be altered by superposition of a uniform hydrostatic tension P.

4. SPECIAL RESPONSE LAWS

The displacement solution for volume preserving spherical expansion is given by eqn (3.1). The stress solution for uniform internal pressurization of a hollow sphere of homogeneous, isotropic, incompressible elastic-plastic (or nonlinearly elastic) material is given, formally, by eqns (3.12)–(3.14). If the uniaxial stress response function is partitioned into an elastic response and a plastic response, as in eqn (2.2), then the stress solution is given by eqns (3.12) and (3.14), supplemented by eqns (3.20)–(3.22).

Of course, this solution is no more complete than that given in eqns (3.9). The introduction of the function f, defined by the first order ordinary differential equation (3.13), is strictly a matter of notation. Calculation of the stresses and the pressure-porosity relation, for the general form of the uniaxial stress response function, will involve numerical integrations.

Explicit expressions for the stresses and the pressure-porosity relation may be obtained for special forms of the response law. These forms are such that the integrals in eqns (3.9) may be evaluated in terms of elementary functions or, equivalently, for which the ordinary differential equations (3.13) or (3.22) can be integrated in closed form. This necessitates choosing response functions h having the form

$$\sigma = h(\lambda) = \frac{3}{2}(1 - \lambda^{3/2})f'(\lambda^{3/2}) \tag{4.1}$$

or

$$\sigma = H(\epsilon) = \frac{3}{2}(1 - e^{-3\epsilon/2})f'(e^{-3\epsilon/2}), \tag{4.2}$$

where f' is the derivative of a combination of elementary functions.

The elastic response

A convenient form of the elastic response law is

$$\sigma = 2\mu(1 - e^{-3\epsilon/2}) \qquad (0 \leq \epsilon \leq \hat{\epsilon}), \tag{4.3}$$

where μ is the shear modulus. This reduces to the more conventional linear form $\sigma = 3\mu\epsilon$, for small strains. (Young's modulus $E = 3\mu$ for incompressible materials.) With this choice of the elastic response, the first of equations (3.22) leads to

$$f^e(x) = \frac{4}{3}\mu x. \tag{4.4}$$

The form (4.3) of the elastic response law also leads to

$$\hat{x} = e^{-3\hat{\epsilon}/2} = 1 - \psi; \qquad (\psi = Y/2\mu) \tag{4.5}$$

and eqns (3.15)–(3.17) and (4.5) give

$$\alpha_1 = \frac{\alpha_0 - \psi}{1 - \psi}; \qquad \alpha_2 = \frac{\alpha_0}{1 - \psi}; \qquad c^3 = a_0^3 \frac{\alpha - \alpha_0}{\psi(\alpha_0 - 1)}. \tag{4.6}$$

The plastic response

Various realistic compressive strain hardening laws have the form (4.2). For each such law, the function $f^p(x)$ is found by integrating the second ordinary differential equation (3.22) with initial condition obtained from eqns (3.21) and (4.4), i.e.

$$f^p(\hat{x}) = \frac{4}{3}\mu\hat{x}. \tag{4.7}$$

1. *A saturation hardening law.* The plastic response law

$$\begin{aligned} \sigma &= Y_1\{1 + \kappa(1 - e^{-3\epsilon/2})^\gamma\} \qquad (\hat{\epsilon} \leq \epsilon < \infty) \\ Y_1 &= Y/(1 + \kappa\psi^\gamma), \end{aligned} \tag{4.8}$$

where κ and γ are positive constants, describes saturation hardening with initial strength Y and ultimate strength Y_∞, given by

$$Y_\infty = Y\frac{1 + \kappa}{1 + \kappa\psi^\gamma}. \tag{4.9}$$

The hardening rate $d\sigma/d\epsilon$ decreases monotonically for $\gamma \leq 1$, and for $\gamma > 1$ it increases up to strain $\epsilon_1 = 2/3 \ln \gamma$, and then decreases. The jump in $d\sigma/d\epsilon$ at the initial strain $\hat{\epsilon}$ is always negative for $\gamma \leq 1$, and it is nonpositive for $\gamma > 1$ provided

$$\kappa \leq 1/\psi^\gamma(\gamma - 1). \tag{4.10}$$

Elastic-plastic response curves from eqns (4.3) and (4.8), with $\psi = .005$, $\kappa = 4$, and various values of γ, are shown in Fig. 1a.

The second equation (3.22) and eqn (4.8) give

$$\frac{d}{dx} f^p(x) = \frac{2}{3} Y_1 \left\{\frac{1}{1 - x} + \kappa(1 - x)^{\gamma - 1}\right\} \tag{4.11}$$

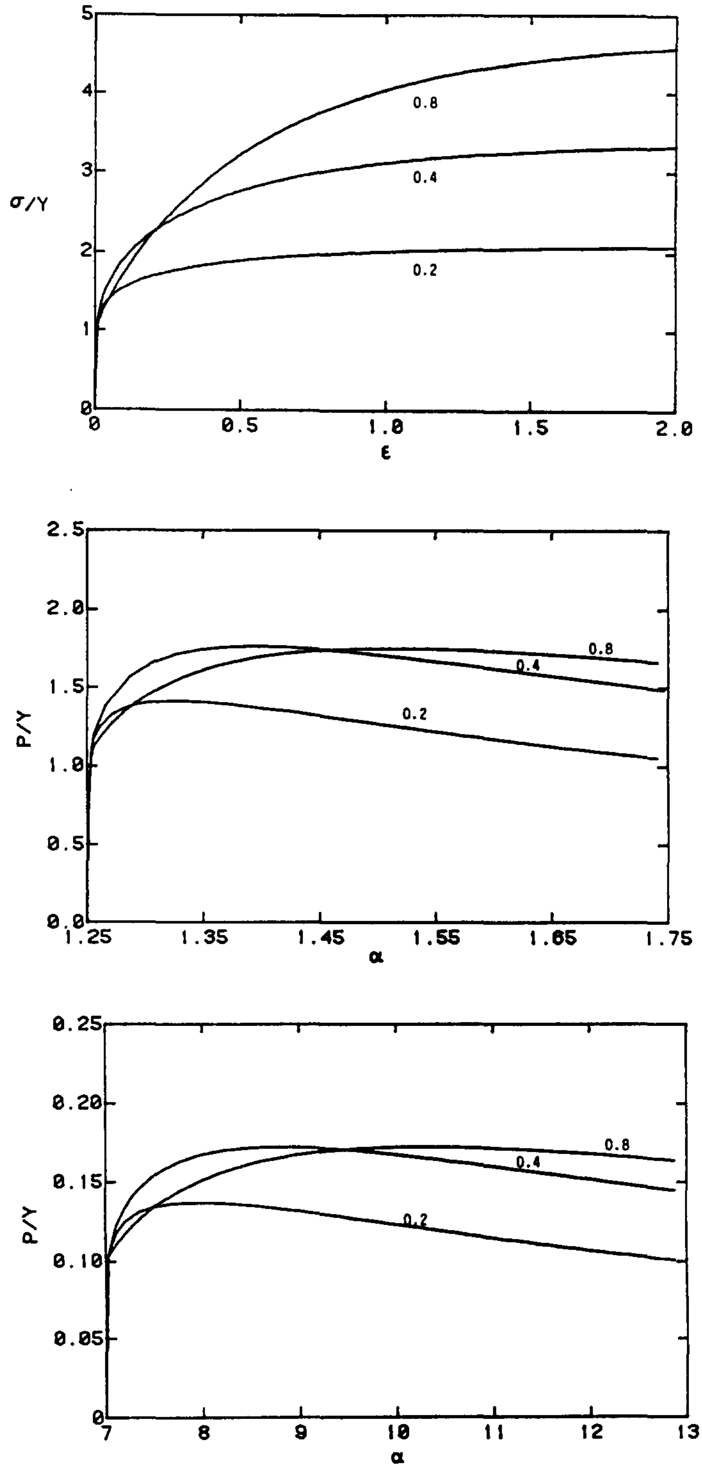


Fig. 1. Stress-strain curves and pressure response curves in spherical expansion for the material defined by equations (5.1). (a) Stress-strain curves from eqns (5.1), with $\psi = Y/2\mu = .005$, $\kappa = 4$, and $\gamma = 0.2$, 0.4, and 0.8. (b) Pressure response curves, from eqn (5.2), for a hollow sphere with initial porosity $\alpha_0 = 1.25$. (c) Pressure response curves, from eqn (5.2), for a hollow sphere with $\alpha_0 = 7$. (d) Pressure response curves, from eqns (8.5) and (8.7), for a spherical cavity in an infinite medium.

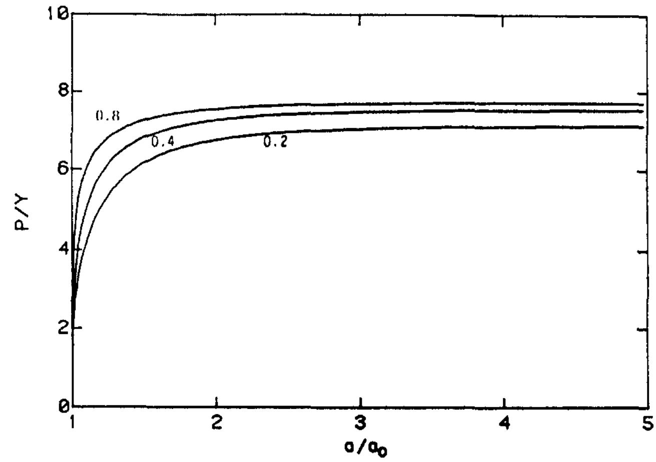


Fig. 1. *Continued.*

and eqns (4.5), (4.7), and (4.11) give

$$
\begin{aligned}
f^p(x) &= -\frac{2}{3} Y_1 \left\{ \ln(1-x) + \frac{\kappa}{\gamma}(1-x)^\gamma \right\} + C_1 \\
C_1 &= \frac{2}{3} Y_1 \{\ln\psi + \kappa\psi^\gamma/\gamma\} + \frac{4}{3}\mu(1-\psi).
\end{aligned}
\tag{4.12}
$$

2. *A "pseudo-exponential" hardening law.* The plastic response law

$$
\begin{aligned}
\sigma &= Y_2\{1 + \kappa e^{\gamma\epsilon}(1 - e^{-3\epsilon/2})\} \qquad (\hat{\epsilon} \leqslant \epsilon < \infty) \\
Y_2 &= Y/\{1 + \kappa\psi/(1-\psi)^{2\gamma/3}\},
\end{aligned}
\tag{4.13}
$$

where κ and γ are positive constants, describes hardening with unbounded ultimate strength. The hardening rate increases monotonically for $\gamma \geqslant 3/4$, and for $\gamma < 3/4$ it decreases up to strain $\epsilon_2 = 4/3 \ln(3/2\gamma - 1)$ and then increases. For $\gamma = 3/4$, eqn (4.13) reduces to

$$
\sigma = Y_2(1 + 2\kappa \sinh 3\epsilon/4) \qquad (\hat{\epsilon} \leqslant \epsilon < \infty). \tag{4.14}
$$

Elastic-plastic response curves from eqns (4.3) and (4.13), with $\psi = .005$, $\kappa = 1$, and various values of γ, are shown in Fig. 2a.

The second equation (3.22) and eqn (4.13) give

$$
\frac{d}{dx} f^p(x) = \frac{2}{3} Y_2 \left(\frac{1}{1-x} + \kappa x^{-2\gamma/3} \right) \tag{4.15}
$$

and eqns (4.5), (4.7), and (4.15) give

$$
\begin{aligned}
f^p(x) &= \frac{2}{3} Y_2 \left\{ -\ln(1-x) + \frac{\kappa}{1 - 2\gamma/3} x^{1-2\gamma/3} \right\} + C_2 \\
C_2 &= \frac{2}{3} Y_2 \left\{ \ln\psi - \frac{\kappa(1-\psi)^{1-2\gamma/3}}{1 - 2\gamma/3} \right\} + \frac{4}{3}\mu(1-\psi).
\end{aligned}
\tag{4.16}
$$

Equation (4.16) does not apply in the special case $\gamma = 3/2$. In this case, the hardening law (4.13) describes exponential hardening. It is most conveniently treated as a special case $\beta = 1$ of the next hardening law (3.17).

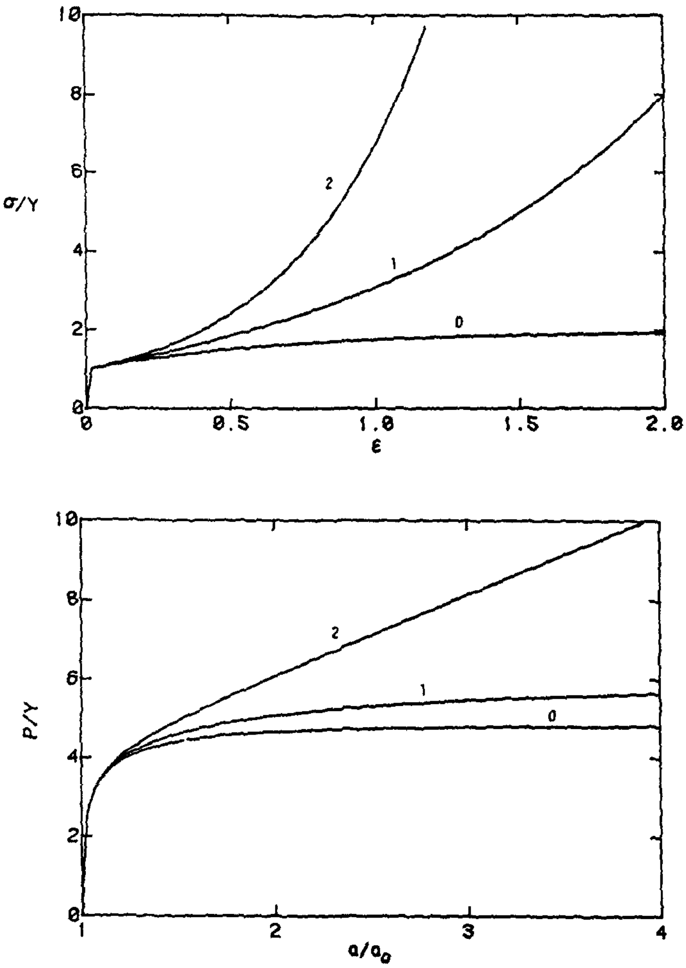


Fig. 2. (a) Stress-strain curves from eqns (4.3) and (4.13), with ψ = .005, κ = 1, and γ = 0, 1, and 2. (b) Corresponding pressure response cruves, from eqns (8.5) and (8.8), for a spherical cavity in a infinite medium.

3. *Another saturation hardening law.* The plastic response law

$$\begin{aligned} \sigma &= Y_3\{1 + \kappa/(1 - \beta + \beta e^{-3\epsilon/2})\} \qquad (\hat{\epsilon} \leq \epsilon < \infty) \\ Y_3 &= Y(1 - \beta\psi)/(1 - \beta\psi + \kappa), \end{aligned} \tag{4.17}$$

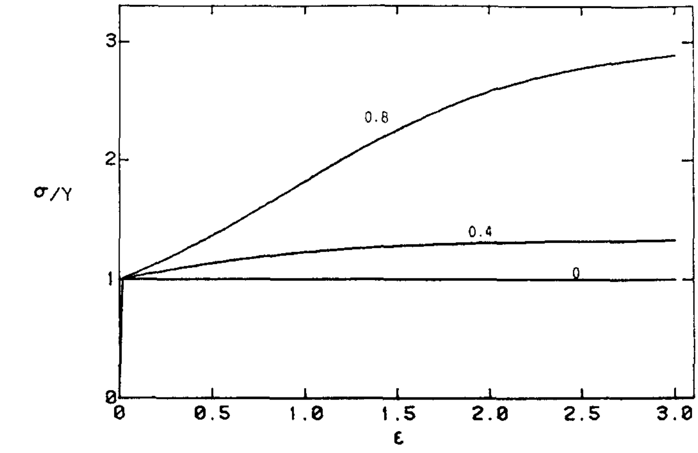
where κ and β are positive constants, with $\beta \leq 1$, describes saturation hardening with ultimate strength

$$Y_\infty = Y\frac{(1 - \beta\psi)(1 - \beta + \kappa)}{(1 - \beta)(1 - \beta\psi + \kappa)}, \tag{4.18}$$

except in the limiting case $\beta = 1$, when it describes exponential hardening with unbounded ultimate strength. The hardening rate decreases monotonically for $\beta \leq 1/2$, and for $\beta > 1/2$ it increases up to strain $\epsilon_3 = 2/3 \ln(\beta/(1 - \beta))$ and then decreases. Elastic-plastic response curves from eqns (4.3) and (4.17), with $\psi = .005$, $\kappa = 1$, and various values of β, are shown in Fig. 3a.

The second equation (3.22) and eqn (4.17) give

$$\frac{d}{dx} f^P(x) = \frac{2}{3} Y_3 \left\{\frac{1}{1 - x} + \frac{\kappa}{(1 - x)(1 - \beta + \beta x)}\right\} \tag{4.19}$$



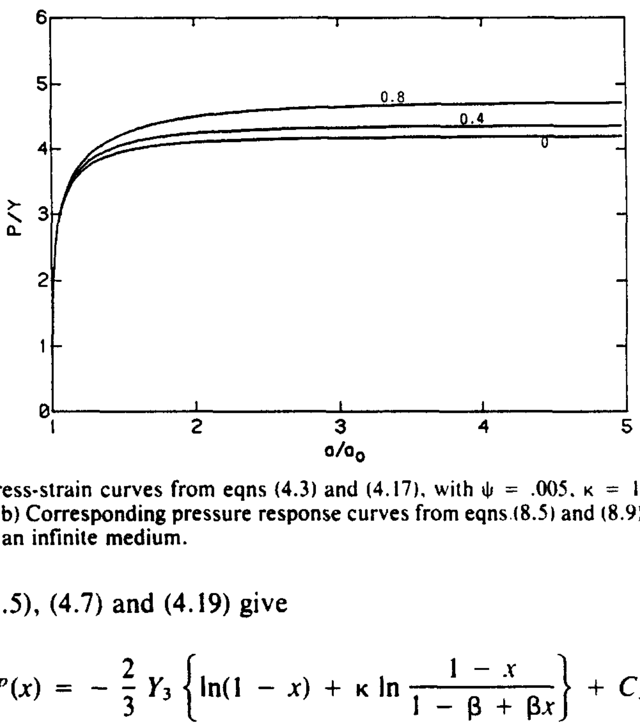


Fig. 3. (a) Stress-strain curves from eqns (4.3) and (4.17), with $\psi = .005$, $\kappa = 1$, and $\beta = 0$, 0.4, and 0.8. (b) Corresponding pressure response curves from eqns (8.5) and (8.9), for a spherical cavity in an infinite medium.

and equations (4.5), (4.7) and (4.19) give

$$\begin{aligned} f^p(x) &= -\frac{2}{3}Y_3\left\{\ln(1-x)+\kappa\ln\frac{1-x}{1-\beta+\beta x}\right\}+C_3 \\ C_3 &= \frac{2}{3}Y_3\left\{\ln\psi+\kappa\ln\frac{\psi}{1-\beta\psi}\right\}+\frac{4}{3}\mu(1-\psi). \end{aligned} \tag{4.20}$$

The response law (4.17) is a particular case of the more general plastic response law

$$\sigma = A\{1+\kappa/(1-\beta+\beta e^{-3\epsilon/2})^{\gamma}\}, \tag{4.21}$$

which allows integration of the second equation (3.22) in closed form for integer or inverse values $\gamma = n$ or $1/n$ $(n = 1, 2, \ldots)$.

4. *"Pseudo-linear" hardening laws.* The plastic response law

$$\begin{aligned} \sigma &= Y_4\{1+\kappa\epsilon(1-e^{-3\epsilon/2})\} \qquad (\hat{\epsilon} \leq \epsilon < \infty) \\ Y_4 &= Y\bigg/\left\{1-\frac{2}{3}\kappa\psi\ln(1-\psi)\right\}, \end{aligned} \tag{4.22}$$

where κ is a positive constant, describes hardening with unbounded ultimate strength which approaches linear strain hardening at higher strain. The hardening rate increases up to strain $\epsilon_4 = 4/3$ and then decreases. Elastic-plastic response curves from eqns (4.3) and (4.22), with $\psi = .005$ and various values of κ, are shown in Fig. 4a.

The second equation (3.2) and eqn (4.22) give

$$\frac{d}{dx} f^p(x) = \frac{2}{3} Y_4 \left\{ \frac{1}{1-x} - \frac{2}{3}\kappa \ln x \right\} \tag{4.23}$$

and eqns (4.5), (4.7), and (4.23) give

$$\begin{aligned} f^p(x) &= -\frac{2}{3} Y_4 \left\{ \ln(1-x) + \frac{2}{3}\kappa(x \ln x - x) \right\} + C_4 \\ C_4 &= \frac{2}{3} Y_4 \left[\ln \psi + \frac{2}{3}\kappa\{(1-\psi)\ln(1-\psi) - 1 + \psi\} \right] + \frac{4}{3}\mu(1-\psi). \end{aligned} \tag{4.24}$$

Plastic response laws of the form

$$\sigma = A\{1 + \kappa\epsilon(1 - e^{-3n\epsilon/2})\} \qquad (n = 2, 3, \ldots) \tag{4.25}$$

provide better approximations to linear strain hardening as n increases, and they also allow solution of the second equation (3.22) in closed form. However, the expressions for $f^p(x)$ become increasingly complicated as n increases. The pseudo-exponential hard-

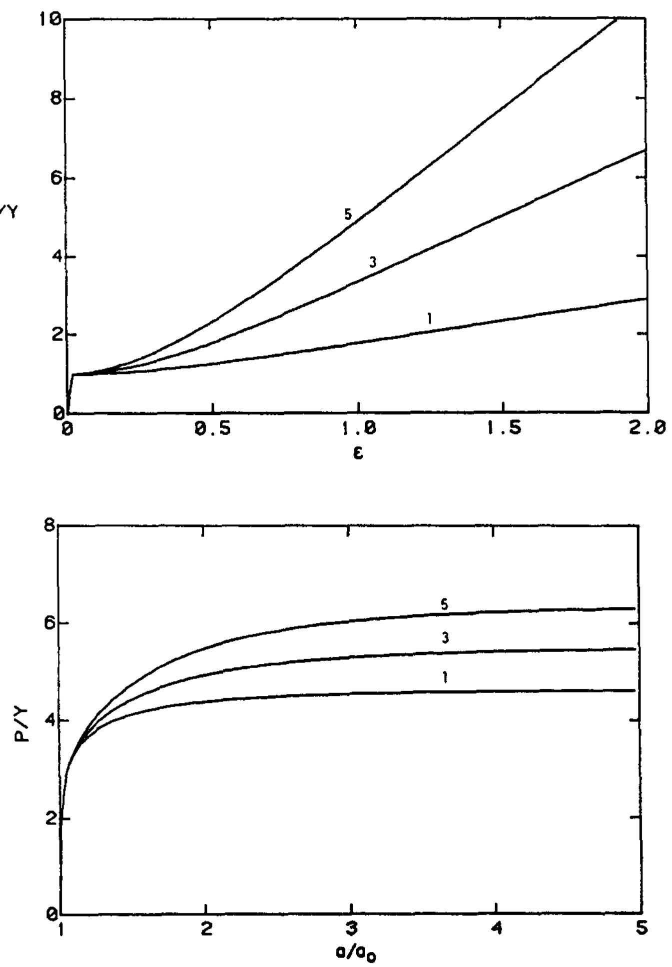


Fig. 4. (a) Stress-strain curves from eqns (4.3) and (4.22), with $\psi = .005$ and $\kappa = 1$, 3, and 5. (b) Corresponding pressure response curves, from eqns (8.5) and (8.10), for a spherical cavity in an infinite medium.

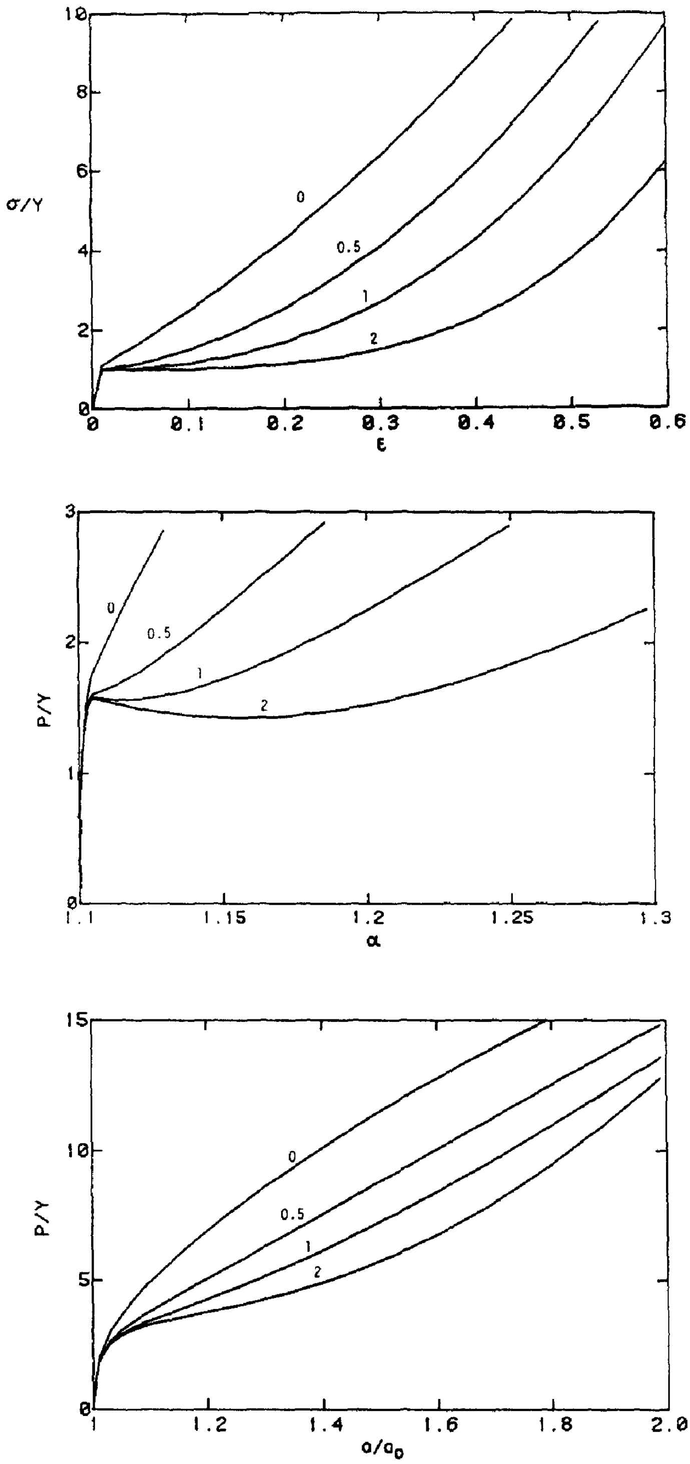


Fig. 5. Stress-strain curves and pressure response curves in spherical expansion for the material defined by eqns (4.3) and (4.27). (a) Stress-strain curves for $\psi = .005$, $\kappa = 10$, and $\gamma = 0, .5, 1$, and 2. (b) Pressure response curves, from eqn (5.11), for a hollow sphere with initial porosity $\alpha_0 = 1.1$. (c) Pressure response curves, from eqns (8.5) and (8.11), for a spherical cavity in an infinite medium.

ening law (4.13) may be generalized in a similar manner to more closely approximate exponential hardening. Exponential hardening laws of the form

$$\sigma = A(1 + \kappa e^{\gamma\epsilon}) \tag{4.26}$$

allow solution of the second equation (3.22) in closed form for values $\gamma = 3n/2$ or $3/2n$ ($n = 1, 2, \ldots$). Again, the expressions for $f^p(x)$ become increasingly complicated as n increases.

5. *Power law—exponential hardening.* The plastic response law

$$\begin{aligned} \sigma &= Y_5\{1 + \kappa\epsilon^{\gamma}(e^{3\epsilon/2} - 1)\} \\ Y_5 &= Y \Big/ \left\{1 + \kappa\left(\frac{2}{3}\ln\frac{1}{1-\psi}\right)^{\gamma}\frac{\psi}{1-\psi}\right\}, \end{aligned} \tag{4.27}$$

with κ and γ positive, describes hardening with monotonically increasing hardening rate and unbounded ultimate strength. Elastic-plastic response curves from eqns (4.3) and (4.26), with $\psi = .005$, $\kappa = 10$, and various values of γ, are shown in Fig. 5a.

The second equation (3.22) and eqn (4.27) give

$$\frac{d}{dx} f^p(x) = \frac{2}{3} Y_5 \left\{\frac{1}{1-x} + \kappa\left(-\frac{2}{3}\ln x\right)^{\gamma}\frac{1}{x}\right\} \tag{4.28}$$

and eqns (4.3), (4.7), and (4.27) give

$$\begin{aligned} f^p(x) &= -\frac{2}{3} Y_5 \left\{\ln(1-x) + \frac{\kappa}{1+\gamma}\left(\frac{2}{3}\right)^{\gamma}(-\ln x)^{1+\gamma}\right\} + C_5 \\ C_5 &= \frac{2}{3} Y_5 \left\{\ln\psi + \frac{\kappa}{1+\gamma}\left(\frac{2}{3}\right)^{\gamma}\left(\ln\frac{1}{1-\psi}\right)^{1+\gamma}\right\} + \frac{4}{3}\mu(1-\psi). \end{aligned} \tag{4.29}$$

5. SPHERICAL EXPANSION FOR SPECIAL MATERIALS

The response laws treated in the previous section do not, of course, exhaust the possibilities of finding solutions in closed form for the problem of pressurization of hollow spheres of elastic-plastic hardening materials. In particular, other useful hardening laws may be found by linear combinations of those just treated. However, the set of plastic response laws in eqns (4.8), (4.13), (4.17), (4.22) and (4.27) provide a considerable flexibility for fitting actual strain hardening data over a fairly extended range of strain.

The complete solution of the problem of spherical expansion for these materials is given by eqns (3.14) and (3.23), with x_a, x_b, and x_r given by eqns (3.10) and (3.12), $f^e(x)$ by eqn (4.4) and $f^p(x)$ by eqns (4.12), (4.16), (4.20), (4.24), or (4.29).

For the saturation hardening model of eqns (4.3) and (4.8), i.e.

$$\sigma = \begin{cases} 2\mu(1 - e^{-3\epsilon/2}) & (0 \leq \epsilon \leq \hat{\epsilon}) \\[2ex] \dfrac{Y}{1 + \kappa\psi^{\gamma}}\{1 + \kappa(1 - e^{-3\epsilon/2})^{\gamma}\} & (\hat{\epsilon} \leq \epsilon < \infty), \end{cases} \tag{5.1}$$

the solution for the pressure is given by eqns (3.23), (4.4) and (4.12) as

$$
\begin{cases}
\dfrac{4}{3}\mu\dfrac{\alpha-\alpha_0}{\alpha(\alpha-1)} & (\alpha_0 \leqslant \alpha \leqslant \alpha_1) \\[2ex]
\dfrac{2}{3}Y\left[\dfrac{1}{1+\kappa\psi^\gamma}\left\{\ln\dfrac{\alpha-\alpha_0}{\psi(\alpha-1)}+\dfrac{\kappa}{\gamma}\left[\left(\dfrac{\alpha-\alpha_0}{\alpha-1}\right)^\gamma-\psi^\gamma\right]\right\}+1-\dfrac{\alpha-\alpha_0}{\psi\alpha}\right] & \\
 & (\alpha_1 \leqslant \alpha \leqslant \alpha_2) \\[2ex]
\dfrac{2}{3}Y\dfrac{1}{1+\kappa\psi^\gamma}\left[\ln\dfrac{\alpha}{\alpha-1}+\dfrac{\kappa}{\gamma}\left\{\left(\dfrac{\alpha-\alpha_0}{\alpha-1}\right)^\gamma-\left(\dfrac{\alpha-\alpha_0}{\alpha}\right)^\gamma\right\}\right] & \\
 & (\alpha_2 \leqslant \alpha \leqslant \infty).
\end{cases}
\tag{5.2}
$$

The stress distribution is as follows:

(a) Elastic phase ($\alpha_0 \leqslant \alpha \leqslant \alpha_1$)

$$
\left.
\begin{aligned}
\sigma_{rr} &= \frac{4}{3}\mu(1-a_0^3/a^3)(1-a^3/r^3)-P \\[2ex]
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = \frac{4}{3}\mu(1-a_0^3/a^3)(1+a^3/2r^3)-P
\end{aligned}
\right\}\quad (a \leqslant r \leqslant b). \tag{5.3}
$$

(b) Elastic-plastic phase ($\alpha_1 \leqslant \alpha \leqslant \alpha_2$)

$$
\left.
\begin{aligned}
\sigma_{rr} &= \frac{2}{3}Y\frac{1}{1+\kappa\psi^\gamma}\left\{\ln\frac{r^3}{a^3}+\frac{\kappa}{\gamma}\left(1-\frac{a_0^3}{a^3}\right)\left(1-\frac{a^{3\gamma}}{r^{3\gamma}}\right)\right\}-P \\[2ex]
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = \sigma_{rr}+\frac{Y}{1+\kappa\psi^\gamma}\left\{1+\kappa\left(\frac{a^3-a_0^3}{r^3}\right)^\gamma\right\}
\end{aligned}
\right\}\quad (a \leqslant r \leqslant c) \tag{5.4}
$$

and

$$
\left.
\begin{aligned}
\sigma_{rr} &= \frac{2}{3}Y\frac{1}{1+\kappa\psi^\gamma}\left[\ln\frac{a^3-a_0^3}{\psi a^3}+\frac{\kappa}{\gamma}\left\{\left(1-\frac{a_0^3}{a^3}\right)^\gamma-\psi^\gamma\right\}\right] \\
&\quad +1-\frac{a^3-a_0^3}{\psi r^3}-P \\[2ex]
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = \sigma_{rr}+\frac{2}{3}Y\frac{a^3-a_0^3}{\psi r^3}
\end{aligned}
\right\}\quad (c \leqslant r \leqslant b), \tag{5.5}
$$

with b and c given by eqns (4.6) or

$$
b^3 = a^3 + a_0^3/(\alpha_0-1); \qquad c^3 = (a^3-a_0^3)/\psi. \tag{5.6}
$$

(c) Plastic phase ($\alpha_2 \leqslant \alpha < \infty$)

$$\left.\begin{aligned}
\sigma_{rr} &= \frac{2}{3} Y \frac{1}{1+\kappa\psi^\gamma}\left\{\ln\frac{r^3}{a^3} + \frac{\kappa}{\gamma}\left(1-\frac{a_0^3}{a^3}\right)^\gamma\left(1-\frac{a^{3\gamma}}{r^{3\gamma}}\right)\right\} - P \\
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = \sigma_{rr} + \frac{Y}{1+\kappa\psi^\gamma}\left\{1+\kappa\left(\frac{a^3-a_0^3}{r^3}\right)^\gamma\right\}
\end{aligned}\right\} \quad (a \leqslant r \leqslant b). \tag{5.7}$$

Solutions for the pressure may be read off in a similar way for the hardening laws (4.13), (4.17), (4.22) and (4.27).

For the hardening law (4.13):

$$P = \begin{cases}
\frac{4}{3}\mu\frac{\alpha-\alpha_0}{\alpha(\alpha-1)} & (\alpha_0 \leqslant \alpha \leqslant \alpha_1) \\
\frac{2}{3}Y_2\left[\ln\frac{\alpha-\alpha_0}{\psi(\alpha-1)} - \frac{\kappa}{1-2\gamma/3}\left\{\left(\frac{\alpha_0-1}{\alpha-1}\right)^{1-2\gamma/3}\right.\right. & \\
\left.\left. -(1-\psi)^{1-2\gamma/3}\right\}\right] + \frac{4}{3}\mu(\psi-1+\alpha_0/\alpha) & (\alpha_1 \leqslant \alpha \leqslant \alpha_2) \\
\frac{2}{3}Y_2\left[\ln\frac{\alpha}{\alpha-1} - \frac{\kappa}{1-2\gamma/3}\left\{\left(\frac{\alpha_0-1}{\alpha-1}\right)^{1-2\gamma/3} - \left(\frac{\alpha_0}{\alpha}\right)^{1-2\gamma/3}\right\}\right] & (\alpha_2 \leqslant \alpha \leqslant \infty).
\end{cases} \tag{5.8}$$

For the hardening law (4.17):

$$P = \begin{cases}
\frac{4}{3}\mu\frac{\alpha-\alpha_0}{\alpha(\alpha-1)} & (\alpha_0 \leqslant \alpha \leqslant \alpha_1) \\
\frac{2}{3}Y_3\left[\ln\frac{\alpha-\alpha_0}{\psi(\alpha-1)} + \kappa\ln\frac{(1-\beta\psi)(\alpha-\alpha_0)}{\psi\{\alpha-1-\beta(\alpha-\alpha_0)\}}\right] + \frac{4}{3}\mu(\psi-1+\alpha_0/\alpha) & \\
 & (\alpha_1 \leqslant \alpha \leqslant \alpha_2) \\
\frac{2}{3}Y_3\left\{\ln\frac{\alpha}{\alpha-1} + \kappa\ln\frac{\alpha-\beta(\alpha-\alpha_0)}{\alpha-1-\beta(\alpha-\alpha_0)}\right\} & (\alpha_2 \leqslant \alpha \leqslant \infty).
\end{cases} \tag{5.9}$$

For the hardening law (4.22):

$$P = \begin{cases}
\frac{4}{3}\mu\frac{\alpha-\alpha_0}{\alpha(\alpha-1)} & (\alpha_0 \leqslant \alpha \leqslant \alpha_1) \\
\frac{2}{3}Y_4\left[\ln\frac{\alpha-\alpha_0}{\psi(\alpha-1)} + \frac{2}{3}\kappa\left\{\frac{\alpha_0-1}{\alpha-1}\ln\frac{\alpha_0-1}{\alpha-1}\right.\right. & \\
\qquad\left.\left. -(1-\psi)\ln(1-\psi) - \frac{\alpha_0-1}{\alpha-1} + 1 - \psi\right\}\right] + \frac{4}{3}\mu(\psi-1+\alpha_0/\alpha) & (\alpha_1 \leqslant \alpha \leqslant \alpha_2) \\
\frac{2}{3}Y_4\left[\ln\frac{\alpha}{\alpha-1} + \frac{2}{3}\kappa\left\{\frac{\alpha_0-1}{\alpha-1}\ln\frac{\alpha_0-1}{\alpha-1} - \frac{\alpha_0}{\alpha}\ln\frac{\alpha_0}{\alpha} + \frac{\alpha-\alpha_0}{\alpha(\alpha-1)}\right\}\right] & (\alpha_2 \leqslant \alpha < \infty)
\end{cases} \tag{5.10}$$

For the hardening law (4.27):

$$P = \begin{cases} \frac{4}{3}\mu\frac{\alpha - \alpha_0}{\alpha(\alpha - 1)} & (\alpha_0 \leq \alpha \leq \alpha_1) \\ \frac{2}{3}Y_5\left[\ln\frac{\alpha - \alpha_0}{\psi(\alpha - 1)} + \frac{\kappa}{1+\gamma}\left(\frac{2}{3}\right)^{\gamma}\left\{\left(\ln\frac{\alpha - 1}{\alpha_0 - 1}\right)^{1+\gamma}\right.\right. & \\ \qquad\qquad \left.\left. - \ln\left(\frac{1}{1-\psi}\right)^{1+\gamma}\right\}\right] + \frac{4}{3}\mu(\psi - 1 + \alpha_0/\alpha) & (\alpha_1 \leq \alpha \leq \alpha_2) \\ \frac{2}{3}Y_5\left[\ln\frac{\alpha}{\alpha - 1} + \frac{\kappa}{1+\gamma}\left(\frac{2}{3}\right)^{\gamma}\left\{\left(\ln\frac{\alpha - 1}{\alpha_0 - 1}\right)^{1+\gamma} - \left(\ln\frac{\alpha}{\alpha_0}\right)^{1+\gamma}\right\}\right] & (\alpha_2 \leq \alpha < \infty) \end{cases} \tag{5.11}$$

The corresponding stress distributions may be read off from eqns (3.14).

Plots of the solutions curves from eqns (5.2) and (5.11) are shown in Fig. 1b and 5b.

6. THE PRESSURE MAXIMUM INSTABILITY

Except for the special case $\beta = 1$ in eqn (5.9), the solutions in eqns (5.2) and (5.8)–(5.10) predict that the pressure attains a maximum value and then decreases to zero. For the solution given in eqns (5.11), the pressure increases monotonically, for sufficiently small values of α_0, or has local maximum and minimum values, for sufficiently large values of α_0; in either case, the pressure eventually increases without bound. In the case of an elastic-perfectly plastic material (eqn (5.2), with $\kappa = 0$) the pressure maximum occurs just before the onset of total yield. Hill[8] suggested that the pressure maximum is an instability point which corresponds to necking or bursting of the sphere. This type of instability may be an important effect at the microlevel, where growth and coalescence of voids in tensile stress fields may lead to rupture or crack propagation.

Conditions for the occurrence of stationary values of the applied load for expansion of hollow spheres and cylinders of incompressible elastic solid material were obtained in a recent paper by Carroll[9]. The instability condition for the spherical expansion problem is equally valid for elastic-plastic materials, with the general hardening law.

The condition for stationary values of the applied pressure is obtained by differentiation of the integral expression in eqns (3.11) and (3.12), i.e.,

$$P = \frac{2}{3}\int_{\alpha_0 - 1/\alpha - 1}^{\alpha_0/\alpha} h(x^{2/3})\frac{dx}{1 - x}. \tag{6.1}$$

Use of Leibnitz' rule gives

$$\frac{dP}{d\alpha} = \frac{2}{3(\alpha - \alpha_0)}\left[\frac{\alpha_0 - 1}{\alpha - 1}h\left\{\left(\frac{\alpha_0 - 1}{\alpha - 1}\right)^{2/3}\right\} - \frac{\alpha_0}{\alpha}h\left\{\left(\frac{\alpha_0}{\alpha}\right)^{2/3}\right\}\right]. \tag{6.2}$$

This may be written

$$\frac{dP}{d\alpha} = \frac{2}{3(\alpha - \alpha_0)}\left\{g\left(\frac{\alpha_0 - 1}{\alpha - 1}\right) - g\left(\frac{\alpha_0}{\alpha}\right)\right\}, \tag{6.3}$$

where the function g is defined by

$$g(x) = xh(x^{2/3}). \tag{6.4}$$

Equation (6.3) gives the condition for stationary values of the pressure as

$$g\left(\frac{\alpha_0 - 1}{\alpha - 1}\right) = g\left(\frac{\alpha_0}{\alpha}\right). \tag{6.5}$$

This may also be written

$$\lambda_a^{3/2}\sigma_a = \lambda_b^{3/2}\sigma_b, \tag{6.6}$$

where σ_a and σ_b denote the values of the compressive stress $\sigma_{\theta\theta} - \sigma_{rr}$ at the inner and outer boundaries.

Differentiation of eqn (6.3) and use of eqn (6.5) gives an expression for $d^2P/d\alpha^2$ at a stationary point $\alpha = \alpha^*$;

$$\left(\frac{d^2P}{d\alpha^2}\right)^* = \frac{2}{3(\alpha - \alpha_0)}\left\{-\frac{\alpha_0 - 1}{(\alpha - 1)^2}g'\left(\frac{\alpha_0 - 1}{\alpha - 1}\right) + \frac{\alpha_0}{\alpha^2}g'\left(\frac{\alpha_0}{\alpha}\right)\right\}. \tag{6.7}$$

It is evident from eqns (6.3) and (6.7) that the qualitative behavior of the pressure is determined by the form of the function g on the interval (0, 1). If the uniaxial compressive stress response function h is monotonic, then the function g may exhibit one of three distinct types of behavior:

(A.) The function g is monotonic in the interval (0, 1). Then eqn (6.5) does not have a real root α in $\alpha_0 < \alpha < \infty$ (see Fig. 6a) and so the pressure in the spherical expansion problem is also monotonic.

(B.) The function g has a maximum value in (0, 1). Then eqn (6.5) has one admissible root α^* (see Fig. 6b). Furthermore, it is evident that

$$g'(\alpha_0/\alpha^*) < 0, \qquad g'\left(\frac{\alpha_0 - 1}{\alpha^* - 1}\right) > 0, \tag{6.8}$$

so that eqn (6.7) gives $(d^2P/d\alpha^2)^* < 0$. It follows that the pressure in the spherical expansion problem increases to a maximum value of $\alpha = \alpha^*$ and then decreases.

(C.) The function g has a local maximum and a local minimum in (0, 1), as shown in Figs. 6c and d. Then the qualitative behavior of the pressure or allround tension in spherical inflation depends on the value of the initial porosity α_0. For sufficiently large values of α_0 (thin-walled spheres), the argument $(\alpha_0 - 1)/(\alpha - 1)$ is only slightly less than α_0/α and eqn (6.5) has two admissible roots, α^* and α^{**}. It is evident from Fig. 6c and eqn (6.7) that the pressure P has a local maximum value at $\alpha = \alpha^*$, decreases to a local minimum value at $\alpha = \alpha^{**}$, and then increases again. There is a critical value of α_0 for which the two roots α^* and α^{**} coincide; in this case, the loading in the spherical expansion problem is monotonic, but there is an inflection point. For smaller values of α_0 (thick-walled spheres), eqn (6.5) does not have an admissible root (Fig. 6d) and the loading curve in spherical expansion is monotonic.

In order to determine which type of behavior will occur for a particular elastic-plastic hardening law, it is convenient to write the function g in terms of the strain ϵ;

$$g(\lambda^{3/2}) = e^{-3\epsilon/2}H(\epsilon) = e^{-3\epsilon/2}\sigma. \tag{6.9}$$

Differentiation shows that the function g increases monotonically as long as

$$\frac{d\sigma}{d\epsilon} \geq \frac{3}{2}\sigma. \tag{6.10}$$

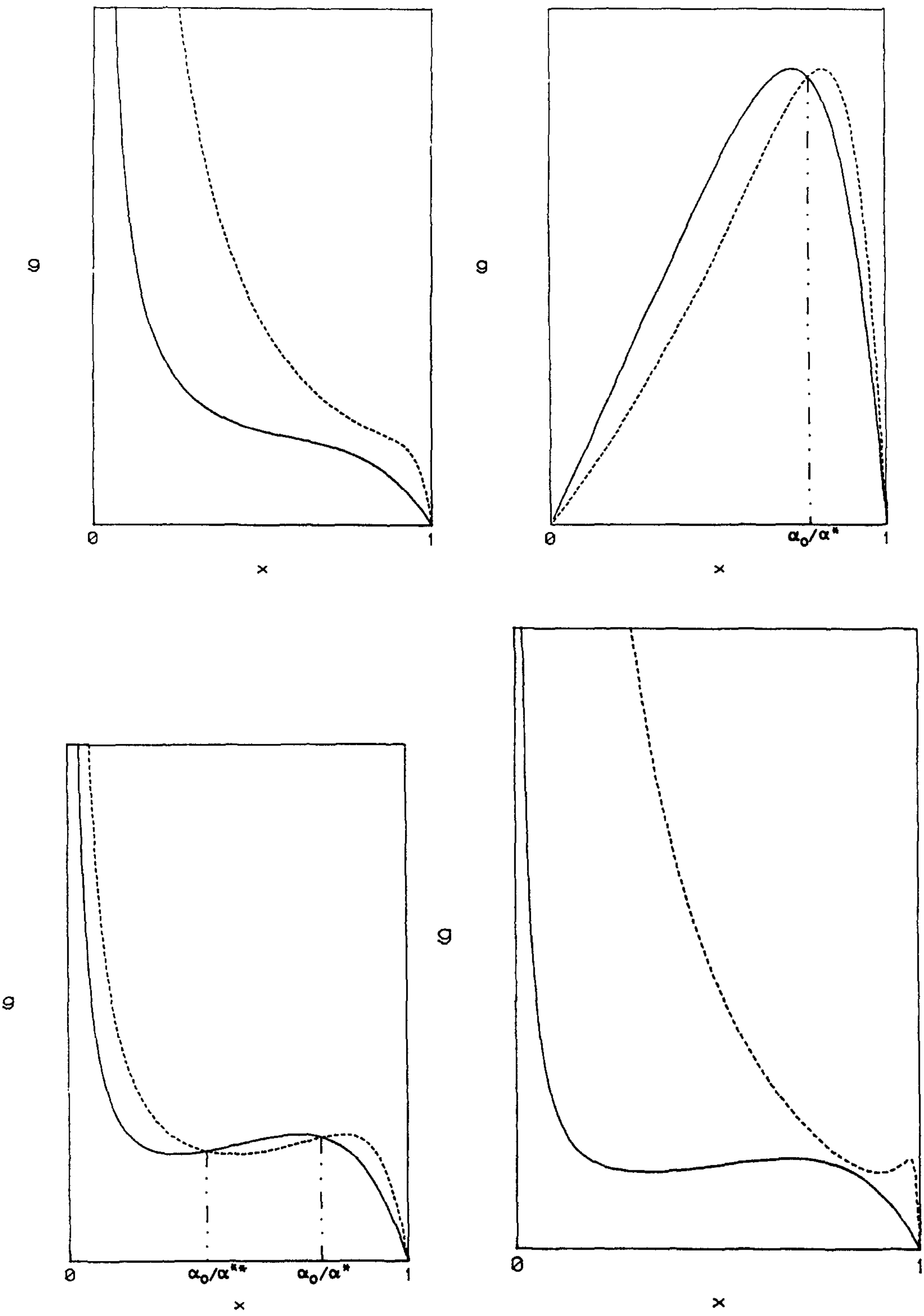


Fig. 6. Qualitative behavior of the function $g(x)$ defined in equation (6.4), and admissible roots of eqn (6.5). The solid curve denotes $g(\alpha_0/\alpha)$ and the dashed curve denotes $g\{(\alpha_0 - 1)/(\alpha - 1)\}$. (a) The function $g(x)$ is monotonic on (0, 1) and there are no admissible roots of eqn (6.5). (b) The function $g(x)$ has a maximum value in (0, 1) and there is one admissible root of eqn (6.5). (c) The function $g(x)$ has local maximum and minimum values in (0, 1) and eqn (6.5) has two admissible roots, for sufficiently large values of α_0. (d) The function $g(x)$ has local maximum and minimum values in (0, 1) and eqn (6.5) has no admissible roots, for sufficiently small values α_0 (>1).

If the material response law is such that the condition (6.10) is met for all compressive strain ($0 \leqslant \epsilon < \infty$), then the behavior in spherical expansion is of type A. If the condition is violated over a range $\epsilon_1 < \epsilon < \infty$, then the behavior is of type B. If the condition is violated for a finite range $\epsilon_1 < \epsilon < \epsilon_2$, then the behavior is of type C.

While elastic-plastic response is typically of type B, all three types of behavior are physically realistic. For example, the widely used Mooney-Rivlin nonlinearly elastic response law admits all three types of behavior, depending on the value of the ratio of the two material constants (Carroll[9]).

The hardening laws (4.8), (4.17), and (4.27) provide a convenient means of analyzing the pressure maximum instability. For elastic-perfectly plastic materials ($\kappa = 0$), it is evident from eqn (6.6) that the pressure maximum cannot occur at or after the onset of total yield, since $\sigma_a = \sigma_b = Y$ for all $\alpha \geqslant \alpha_2$. For $\alpha_1 \leqslant \alpha < \alpha_2$, eqn (6.5) becomes

$$\frac{\alpha_0 - 1}{\alpha - 1}\psi = \frac{\alpha_0}{\alpha}\left(1 - \frac{\alpha_0}{\alpha}\right), \tag{6.11}$$

and this gives a quadratic equation whose admissible root α^* is less than α_2 by an amount of order ψ^2. Thus, the maximum pressure occurs just before the onset of total yield and the maximum pressure P^* is slightly less than the pressure P_2 given by†

$$P_2 = \frac{2}{3} Y \ln \frac{\alpha_2}{\alpha_2 - 1}. \tag{6.12}$$

More generally, the pressure maximum may occur before, at, or after the onset of total yield. This may be seen from the response law (4.8) with $\gamma = 1$, which has the form of the saturation hardening law of Voce[11, 12] and Palm[13]. Assuming that $\alpha^* \geqslant \alpha_2$ and substituting in eqn (6.5) leads to

$$\frac{\alpha_0 - 1}{\alpha - 1} + \frac{\alpha_0}{\alpha} = 1 + \frac{1}{\kappa}. \tag{6.13}$$

It follows that the pressure maximum occurs before, at, or after the onset of total yield, depending on whether $\kappa < \bar{\kappa}$, $\kappa = \bar{\kappa}$, or $\kappa > \bar{\kappa}$, with

$$\frac{1}{\bar{\kappa}} = \frac{\alpha_0 - 1}{\alpha_2 - 1} + \frac{\alpha_0}{\alpha_2} - 1. \tag{6.14}$$

A special limiting case of eqn (4.17), obtained by letting $\beta \to 1$, $\kappa \to \infty$, and $Y_3 \to 0$, gives the exponential hardening model

$$\sigma = \begin{cases} 2\mu(1 - e^{-3\epsilon/2}) & (0 \leqslant \epsilon \leqslant \hat{\epsilon}) \\ (1 - \psi) Y e^{3\epsilon/2} & (\hat{\epsilon} \leqslant \epsilon < \infty). \end{cases} \tag{6.15}$$

For this response law, the function g is constant for $\epsilon \geqslant \hat{\epsilon}$, so that eqn (6.5) is satisfied for all $\alpha \geqslant \alpha_2$. The pressure reaches its maximum value at the onset of total yield and remains constant thereafter. This represents a limiting case between behavior of types A and B. Indeed, a material with exponential plastic response law

$$\sigma = (1 - \psi)^{2\gamma/3} Y e^{\gamma\epsilon} \qquad (\hat{\epsilon} \leqslant \epsilon < \infty) \tag{6.16}$$

exhibits behavior of type A for $\gamma > 3/2$ and behavior of type B $\gamma < 3/2$.

† This was shown by Carroll and Holt[10]. Terms of order ψ^2 were ignored in Hill's[8] analysis, leading to the conclusion that the pressure maximum occurs at the onset of total yield.

Finally, for the response law (4.27), the function g decreases during the earlier part of the plastic range, assuming that

$$\kappa < 1 \Big/ \left\{ \hat{\epsilon}^{\gamma} + \frac{2}{3}\gamma\hat{\epsilon}^{\gamma-1}\psi/(1-\psi) \right\}, \tag{6.17}$$

but it eventually increases again, so that the behavior in spherical expansion is of type C.

7. THE EFFECT OF COMPRESSIBILITY

The assumption of incompressibility has led to a considerable simplification in the analysis of the spherical expansion problem. The symmetry of the problem and the condition that volume be preserved means that the deformation is known *ab initio* so that the problem is reduced to finding the stress distribution. This is no longer the case if the material is compressible. However, the important features of the corresponding solution for compressible materials can be derived quite simply from the incompressible solution, based on two observations.

The first observation is that the change in porosity of a hollow sphere is determined by the deviatoric response of the material. The third equation (3.9) follows directly from the equation of equilibrium and the boundary conditions, without any assumption of incompressibility, and it involves only the deviatoric stress $\sigma = \sigma_{\theta\theta} - \sigma_{rr}$. If the material response is such that hydrostatic and deviatoric effects are not coupled, then σ is determined by the deviatoric strain, and the $P - \alpha$ equation obtained by assuming incompressibility is equally valid for compressible materials. In particular, the maximum pressure P^* and the critical porosity α^* are unchanged.

The $P - \alpha$ equation does not furnish a complete solution of the problem; it merely relates the pressure to the porosity of the sphere or, equivalently, to the ratio a/b of the inner and outer boundaries. The separate behavior of the inner and outer boundaries is then obtained, for incompressible materials, from eqns (3.5). The second observation is that if the hydrostatic response of a compressible material is linearly elastic, then the volume strain of a material region is related to the average pressure in the region through the linearly elastic response law. The average stress $\bar{\sigma}_{ij}$ in a body which is in equilibrium under surface forces only, and occupies a region $\mathcal{R}$ with boundary $\partial\mathcal{R}$, is given by

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_{\mathcal{R}} \sigma_{ij}\,\mathrm{d}V = \frac{1}{V}\int_{\partial\mathcal{R}} t_i x_j\,\mathrm{d}S. \tag{7.1}$$

Here $\bar{\sigma}_{ij}$ and σ_{ij} are the average and local components of Cauchy stress in a rectangular Cartesian system, V is the volume of the region $\mathcal{R}$, and t_i are components of the surface force per unit area at a point on $\partial\mathcal{R}$ with coordinates x_i. The formula (7.1) is obtained simply by using the equations of equilibrium and the divergence theorem.

For compressible materials, one must distinguish between internal pressure and external allround tension, i.e. the boundary conditions (3.7) and (3.24) are no longer equivalent. For internal pressure P, use of eqn (7.1) gives the average pressure $\bar{P}$ as

$$\bar{P} = \frac{a^3}{b^3 - a^3}P, \tag{7.2}$$

and the corresponding expression in the case of external allround tension is

$$\bar{P} = -\frac{b^3}{b^3 - a^3}P. \tag{7.3}$$

In either case,

$$\bar{P} = -K\frac{\Delta V}{V} = K\left\{\frac{b_0^3 - a_0^3}{b^3 - a^3} - 1\right\}, \tag{7.4}$$

where K is the bulk modulus of the material. Equations (7.2) and (7.4) give

$$b^3 - a^3(1 - P/K) = b_0^3 - a_0^3, \tag{7.5}$$

and eqns (7.3) and (7.4) give

$$b^3(1 - P/K) - a^3 = b_0^3 - a_0^3. \tag{7.6}$$

A typical pressure-porosity equation, such as eqn (5.2), may be written as

$$P = \mathcal{P}(\alpha, \alpha_0) = \mathcal{P}\left(\frac{b^3}{b^3 - a^3}, \frac{b_0^3}{b_0^3 - a_0^3}\right). \tag{7.7}$$

Equation (7.5) or (7.6) may now be used to eliminate either a^3 or b^3 from eqn (7.7), thus relating the pressure or allround tension P to the motion of the inner or outer boundaries of a compressible hollow sphere.

8. EXPANSION OF A SPHERICAL CAVITY IN AN INFINITE MEDIUM

In the case of expansion of a spherical cavity of initial radius a_0 in an infinite medium, under uniform internal pressure P, the pressure is given by

$$P = \frac{2}{3}\int_{a_0^3/a^3}^{1} h(x^{2/3})\frac{dx}{1 - x}, \tag{8.1}$$

or

$$P = f(1) - f(a_0^3/a^3). \tag{8.2}$$

Differentiation of eqn (8.1) with respect to a gives

$$\frac{dP}{da} = \frac{2a_0^3}{a(a^3 - a_0^3)} h\left(\frac{a_0^2}{a^2}\right), \tag{8.3}$$

so that the pressure increases monotonically.

The initial response is elastic, and the elastic phase persists until the cavity radius reaches the value a_1, given by the first equations (3.12) and (4.5) as

$$a_1^3 = a_0^3/(1 - \psi). \tag{8.4}$$

The pressure in the elastic phase is given by eqns (4.4) and (8.2) as

$$P = \frac{4}{3}\mu(1 - a_0^3/a^3) \qquad (a_0 \leq a \leq a_1), \tag{8.5}$$

and the pressure P_1 at the onset of yielding is given by eqns (8.4) and (8.5) as

$$P_1 = \frac{2}{3}Y. \tag{8.6}$$

As the pressure is increased, a spherical elastic-plastic interface propagates outward, with radius c given by eqn (5.6). Solutions for the pressure in the elastic-plastic phase ($a_1 \leq a < \infty$), for the five special hardening laws, may be read off from eqns (4.4), (4.12), (4.16), (4.20), (4.24), (4.29), and (8.2).
For the hardening law (4.8):

$$P = \frac{2}{3}Y + \frac{2}{3}Y_1\left[\ln\frac{a^3 - a_0^3}{\psi a^3} + \frac{\kappa}{\gamma}\left\{\left(1 - \frac{a_0^3}{a^3}\right)^{\gamma} - \psi^{\gamma}\right\}\right]. \tag{8.7}$$

For the hardening law (4.13):

$$P = \frac{2}{3}Y + \frac{2}{3}Y_2\left[\ln\frac{a^3 - a_0^3}{\psi a^3} - \frac{\kappa}{1 - 2\gamma/3}\left\{\left(\frac{a_0}{a}\right)^{3-2\gamma} - (1 - \psi)^{1-2\gamma/3}\right\}\right]. \tag{8.8}$$

For the hardening law (4.17):

$$P = \frac{2}{3}Y + \frac{2}{3}Y_3\left[\ln\frac{a^3 - a_0^3}{\psi a^3} + \kappa\ln\frac{(1 - \beta\psi)(a^3 - a_0^3)}{\psi\{a^3 - \beta(a^3 - a_0^3)\}}\right]. \tag{8.9}$$

For the hardening law (4.22):

$$P = \frac{2}{3}Y + \frac{2}{3}Y_4\left[\ln\frac{a^3 - a_0^3}{\psi a^3} + \frac{2}{3}\kappa\left\{\frac{a_0^3}{a^3}\ln\frac{a_0^3}{a^3} - \frac{a_0^3}{a^3} - (1 - \psi)\ln(1 - \psi) + 1 - \psi\right\}\right]. \tag{8.10}$$

For the hardening law (4.27):

$$P = \frac{2}{3}Y + \frac{2}{3}Y_5\left[\ln\frac{a^3 - a_0^3}{\psi a^3} + \frac{\kappa}{1 + \gamma}\left(\frac{2}{3}\right)^{\gamma}\left\{\left(\ln\frac{a^3}{a_0^3}\right)^{1+\gamma} - \left(\ln\frac{1}{1 - \psi}\right)^{1+\gamma}\right\}\right]. \tag{8.11}$$

Plots of the solution curves from eqns (8.5) and (8.7–8.11) are shown in Figs. 1d, 2b, 3b, 4b and 5c. Equations (8.7)–(8.10) predict finite asymptotic values for the pressure, except for the special case $\beta = 1$ in eqn (8.9). In this case, and also for eqn (8.11), the pressure increases without bound.

The stress distribution given in equations (5.3)–(5.7) for the hardening law (4.8) is valid also for pressurization of a spherical cavity in an infinite medium, with the pressure P now given by eqns (8.5) and (8.7). Stress distributions for the other hardening models are obtained in a similar manner.

The solutions given by eqns (8.5) and (8.7)–(8.11) pertain also to expansion of a spherical cavity in an infinite medium due to remote allround tension.

9. THE REQUIRED RANGE OF STRAIN

In using the hardening models in Section 4 to fit experimental data, it is necessary to know the range of strains that is encountered in the spherical expansion problem. The relevant range is that which occurs up to the pressure maximum P^*, at porosity α^*.

Some useful information can be obtained, quite simply, from the kinematics of the problem. The maximum strain occurs at the inner boundary, and it is given by eqns

Table 1. Maximum strain at total yield

a_0/b_0	.05	.1	.2	.3	.4	.5	.6	.7	.8	.9
(ψ = .005) ϵ^+	2.48	1.20	.325	.114	.050	.026	.015	.0097	.0065	.0046
(ψ = .01) ϵ^+	2.94	1.60	.544	.212	.098	.052	.03	.019	.013	.0092

(2.3) and (3.5) as

$$\epsilon_a = \frac{2}{3}\ln\frac{\alpha - 1}{\alpha_0 - 1}. \tag{9.1}$$

Suppose now that strain hardening data is available over the range $0 \leq \epsilon \leq \hat{\epsilon}$. Then the problem of spherical expansion can be solved up to porosity $\hat{\alpha}$, given by

$$\hat{\alpha} = 1 + (\alpha_0 - 1)e^{3\hat{\epsilon}/2}. \tag{9.2}$$

Continuation of the solution to higher values of α requires extrapolation of the available data.

The value α_2 of the porosity at the onset of total yield does not depend on the hardening law. It is given by eqn (4.6) as

$$\alpha_2 = \alpha_0/(1 - \psi) \qquad (\psi = Y/2\mu). \tag{9.3}$$

Substitution from eqn (9.3) in eqn (9.1) shows that solution of the problem of spherical expansion up to total yield requires knowledge of the stress-strain response over the range $0 \leq \epsilon \leq \epsilon^+$, with

$$\epsilon^+ = \frac{2}{3}\ln\frac{\alpha_0 - 1 + \psi}{(\alpha_0 - 1)(1 - \psi)}, \tag{9.4}$$

or

$$\epsilon^+ = \frac{2}{3}\ln\left\{1 + \frac{\psi b_0^3}{(1 - \psi)a_0^3}\right\}. \tag{9.5}$$

This range is illustrated by Table 1, which shows values of ϵ^+ and a_0/b_0 for $\psi = .005$ ($\hat{\epsilon} = .00334$) and for $\psi = .01$ ($\hat{\epsilon} = .0067$):

Finally, for a given hardening law, eqn (6.5) determines the value α^* at the pressure maximum, and substitution in eqn (9.1) gives the corresponding maximum strain.

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